Move patterns of the largest disk

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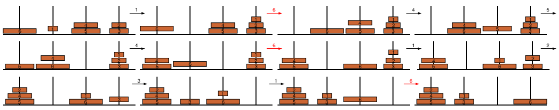
Abstract

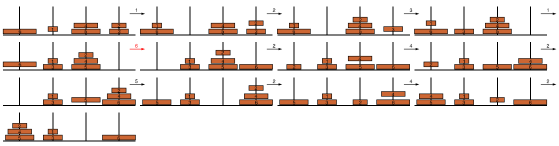
In order to find the optimal solution of a task, i.e. the shortest path between two given states of the multi-peg Tower of Hanoi, it is a first step to determine the moves of the largest disk, as they are most restricted by the divine rule. Surprisingly there are shortest paths with more than one largest disk move (LDM), even in H_3^2. We analyse the existence of shortest paths in the multi-peg Tower of Hanoi with a given pattern of LDMs using numerical and analytical methods.

S. Aumann, A. M. Hinz, C. Petr: Moves of the largest disc in the multi-peg Tower of Hanoi, in preparation, 2009

Example

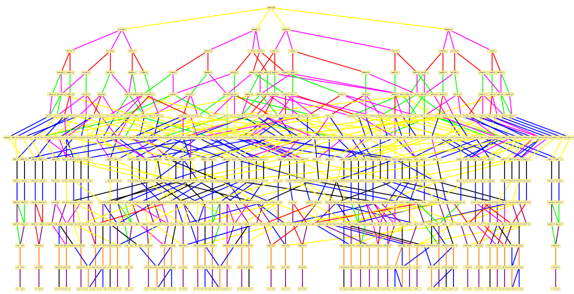


A shortest path in H_4^6 with 3 LDMs (red)



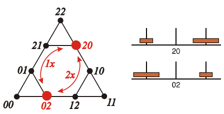
in comparison: a longer path between the same states with 1 LDM; in fact, the shortest path is unique.

Visualizing distances and shortest paths

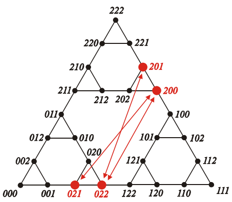


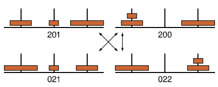
A distance layered graph from a perfect state in H_5^9 to all states where a LDM is possible for the first time.

3 pegs



A task in H_3^2 with 2 shortest paths: one has 1 LDM and the other 2.





3 tasks in H_3^3 where 2 LDMs are neccessary.

A quantitative overview gives the following.

Theorem For $n \in \mathbb{N}$ the 9^{n-1} tasks (s,t) from H_3^n with $s_n = 0, t_n = 2$ decompose into

- $\frac{1}{14}\left(13 \cdot 9^{n-1} + 2^{n-1} - 7x_{n-1}\right)$ with exactly 1 LDM
- $\frac{1}{14}\left(9^{n-1} + 2^{n-1} - 7x_{n-1}\right)$ with 2 neccessary LDMs
- $x_{n-1} = \sum_{i=1}^{2^{n-1}-1} s_i s_{n-i+1}$ with 1 or 2 occuring LDMs

where $x_n = \frac{1}{\sqrt{17}}\left(\Theta_+^n - \Theta_-^n\right)$, $\Theta_\pm = \frac{1}{2}\left(5 \pm \sqrt{17}\right)$ and s_i is Stern's diatomic sequence.

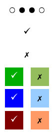
Some results ...

- ⋆ The largest disk moves at most $p-1$ times in general.
- ⋆ The largest disk moves at most once if the initial or the goal state is perfect, i.e. all disks lie on one peg.
- ⋆ The largest disk moves at most twice if there are more pegs than disks.
- ⋆ If $n \geq p(p-2)$, $p-1$ largest disk moves are neccessary for some tasks.

... and conjectures

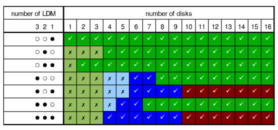
- ⋆ There are tasks whose shortest paths may have $p-1$ LDMs, if $n \geq 2(p-2)$.
- ⋆ There are tasks whose shortest paths must have $p-1$ LDMs, if $n \geq 3(p-2)$.

More than 3 pegs

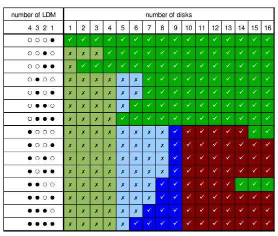


Pattern of LDMs, e.g. 2 and 3, but not 1 or 4 LDMs
There exist some tasks with the given pattern.
There do not exist any tasks with the given pattern.
The result is mathematically proved.
The result is numerically computed.
The result is just conjectured.

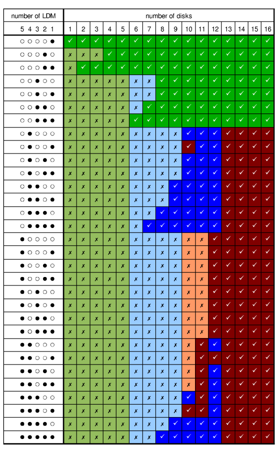
4 pegs



5 pegs



6 pegs



7 pegs