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ALGORITHM FOR THE
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PROBLEM

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On the Frame-Stewart algorithm for the multi-peg Tower of Hanoi problem

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Abstract

It is proved that seven different approaches to the multi-peg Tower of Hanoi problem are all equivalent. Among them the classical approaches of Stewart and Frame from 1941 can be found.

1 Introduction

The Tower of Hanoi problem(s) presents an unusual phenomenon in the mathematical and computer science literature. On one side it was extensively studied by many authors, Stockmeyer's survey [14] lists 207 relevant references, not included articles in psychological journals and textbooks in discrete mathematics. On the other hand, many papers only rediscover known results, "prove" results under wrong assumptions, or claim to solve the general multi-peg problem by solving a particular recursion for which it is not known whether it is really an optimal one. For instance, Stockmeyer [13] lists 13 (!) references which more or less only rediscover contributions of Frame [6] and Stewart [12] from 1941. Another concrete example of several wrong approaches is the false assumption that the largest disc moves at most once in a shortest path between two regular states of the three peg problem. This confusion is nicely described and clarified by Hinz in [8]. We also refer to a nice semi-survey of Poole [10].

Probably the greatest challenge related to the Towers of Hanoi is given by increasing the number of pegs to more than three. In 1908 Dudeney [4] posed

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the problem for four pegs and called it the *Reve's Puzzle*. Then, in 1941, two nowadays famous “solutions” for any number of pegs followed, one due to Frame [6] and the other to Stewart [12]. Most researchers believe that the solutions proposed by Frame and Stewart are optimal. Interestingly, many also feel no need for proving this fact, even if it was pointed out already in 1941 by the problems editor of the American Mathematical Monthly [5] that their proofs of optimality only applies to algorithms of a certain scheme! For instance, in the 'seventies additional “solutions” followed [11,1]. Then in the early 'eighties Wood [15] and Cull and Ecklund [3] correctly pointed out that the problem is still open. However, even this was not enough to prevent new “solutions”. For instance, very recently [9] one can read in the abstract: “This paper solves completely the generalized p -peg Tower of Hanoi problem when $p \geq 4$ ”. In reality the author restricts himself to the Stewart's recursive scheme and on this particular scheme applies the dynamic programming.

In literature the solutions of Frame and Stewart are presented in different forms. In fact, even if the solutions of Frame and Stewart are “essentially the same”, cf. [13], we could find no published rigorous proof of this fact. (It is true, though, that for the Reve's Puzzle the solutions are easily seen to be the same.) Since in the context of the Tower of Hanoi problems there has already been several obvious (but wrong) facts around, we feel justified to clarify this point of view. More precisely, we present seven² different approaches and prove that all of them are equivalent. Thus, after proving this result one can really speak about *the* presumed optimal solution.

The rest of the paper is organized as follows. In the next section, we introduce the following functions:

$F(n, p)$... reflects Frame's algorithm;

$F'(n, p)$... same as $F(n, p)$, but no monotonicity is required;

$S(n, p)$... reflects Stewart's algorithm;

$A'(n, p)$... reflects an algorithm taking into account all partition;

$A(n, p)$... same as $A'(n, p)$, but monotonicity is required.

In Section 3 we continue with a proof that $A' = F' = S$. Then we introduce an explicit formula

$X(n, p)$... which is (essentially) a sum of powers of 2,

and prove an equivalent expression for it. In Section 5 we demonstrate that $X = S$. In the final section we show that F and A coincide with all the rest

² Or maybe six — it depends on whether two different presentations of $X(n, p)$ are counted separately or not.

which enables us to state the result of this paper:

$$A(n, p) = A'(n, p) = F(n, p) = F'(n, p) = S(n, p) = X(n, p).$$

2 Recursive definitions

Let $M(n, p)$ be the minimum number of moves required to solve the Tower of Hanoi problem with n discs and p pegs.

Only for $p \geq 4$ pegs and $n \geq p$ discs the problem considered is nontrivial, i.e. it has been proved that $M(n, 3) = 2^n - 1$, and that $M(n, p) = 2n - 1$ for $n < p$. Therefore, in trivial range $p = 3$ or $n < p$ we shall define all the functions treated here to be defined as $M(n, p)$. For technical reasons it is useful to take $M(1, 2) = 1$ and $M(n, 2) = 0$, for $n \geq 2$.

Definition 2.1 *If $p = 2, 3$ or $n < p$*

$$F(n, p) = F'(n, p) = A(n, p) = A'(n, p) = S(n, p) = M(n, p).$$

All five different definitions for recursive calculations of presumed optimal solution will be now given only outside the trivial range, for $n, p \in \mathbb{N}$ satisfying $p \geq 4$ pegs and $n \geq p$, taking the values from the trivial range as initial values, when required. Also, when proving results about these functions and about explicit formulas, discussed in Sections 4 and 5, some care will be taken about the trivial range.

The following definition reflects the **F**rame's algorithm for the multi-peg Tower of Hanoi problem, but it differs from the original definition, since it does not require partitions of n to be monotone.

Definition 2.2 *For $p \geq 4$ and $n \geq p$ let $F'(n, p)$ be the minimum of*

$$\{2F'(n_1, p) + 2F'(n_2, p-1) + \cdots + 2F'(n_{p-2}, 3) + 1 \mid n_1 + \cdots + n_{p-2} + 1 = n\},$$

where $n_i \in \mathbb{N}$.

The original **F**rame's definition was essentially the following:

Definition 2.3 *For $p \geq 4$ and $n \geq p$ let $F(n, p)$ be the minimum of*

$$\{2F(n_1, p) + 2F(n_2, p-1) + \cdots + 2F(n_{p-2}, 3) + 1 \mid n_1 + \cdots + n_{p-2} + 1 = n, n_1 \geq n_2 \geq \cdots \geq n_{p-2}\},$$

where $n_i \in \mathbb{N}$.

In Figure 1 we present strategy for disk moves corresponding to $F(n, p)$ or $F'(n, p)$.

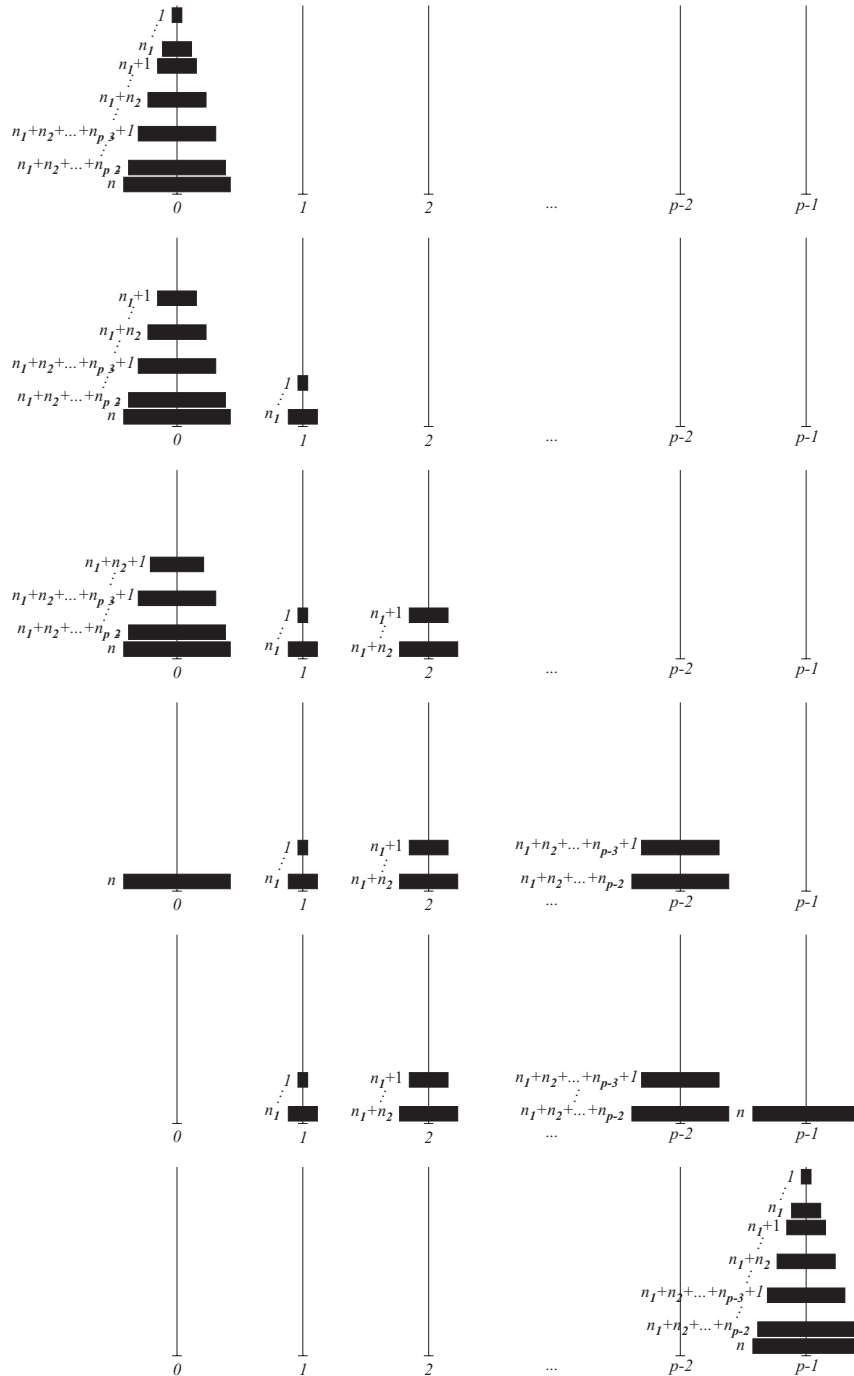


Fig. 1. Disk moves corresponding to $F(n, p)$ or $F'(n, p)$

First we have all n disks on the source peg 0 . We divide $n - 1$ smallest disks into $p - 2$ groups of disks of consecutive size. Top n_1 smallest disks are moved from the source peg 0 to the auxiliary peg 1 using all p pegs. Next n_2 smallest disks are moved from the source peg 0 to the auxiliary peg 2 using $p - 1$ pegs.

We continue removing groups of disks from the source peg to the auxiliary pegs using in each step one peg less than in the previous one. At the end the last group of disks is moved from the source peg 0 to the auxiliary peg $p - 2$ using the remaining 3 admissible pegs. The last (n -th) disk is then moved from the source peg 0 to the destination peg $p - 1$. After that we move all groups of disks to the destination peg, taking them in reverse order.

We now introduce the function $S(n, p)$ which corresponds to the Stewart's algorithm for the multi-peg Tower of Hanoi problem.

Definition 2.4 For $p \geq 4$ and $n \geq p$ let

$$S(n, p) = \min\{2S(n_1, p) + S(n_2, p - 1) \mid n_1 + n_2 = n\},$$

where $n_1, n_2 \in \mathbb{N}$.

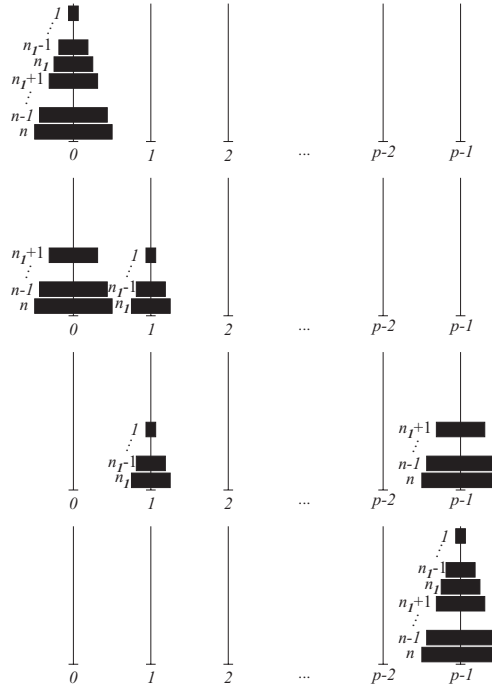


Fig. 2. Disk moves corresponding to $S(n, p)$

In Figure 2 we present strategy for disk moves corresponding to $S(n, p)$.

First all n disks are on the source peg 0. We divide disks into two groups, the first consisting of smallest n_1 disks and the second consisting of the remaining $n - n_1$ disks. Top n_1 disks are moved from the source peg 0 to the auxiliary peg 1 using all p pegs. The remaining $n - n_1$ disks are then moved from the source peg 0 to the destination peg $p - 1$ using $p - 1$ pegs (all of them but peg 1). Finally disks from peg 1 are moved to the destination peg $p - 1$ using all p pegs.

By simple examples it can be show that $S(n, p)$ does not have a monotone counterpart (i.e. it gives a different function).

Next we introduce an algorithm taking into account **all** partitions:

Definition 2.5 For $p \geq 4$ and $n \geq p$ let $A'(n, p)$ be defined as the minimum of

$$\begin{aligned} & \{2A'(n_1, p) + 2A'(n_2, p - 1) + \cdots + 2A'(n_{p-2}, 3) + 1 \mid \\ & \qquad n_1 + \cdots + n_{p-2} + 1 = n\} \cup \\ & \{2A'(n_1, p) + 2A'(n_2, p - 1) + \cdots + 2A'(n_{p-3}, 4) + A'(n_{p-2}, 3) \mid \\ & \qquad n_1 + \cdots + n_{p-2} = n\} \cup \cdots \cup \\ & \{2A'(n_1, p) + A'(n_2, p - 1) \mid n_1 + n_2 = n\}, \end{aligned}$$

where $n_i \in \mathbb{N}$.

The monotone version of $A'(n, p)$ is as follows:

Definition 2.6 For $p \geq 4$ and $n \geq p$ let $A(n, p)$ be defined as the minimum of

$$\begin{aligned} & \{2A(n_1, p) + 2A(n_2, p - 1) + \cdots + 2A(n_{p-2}, 3) + 1 \mid \\ & \qquad n_1 + \cdots + n_{p-2} + 1 = n, n_1 \geq n_2 \geq \cdots\} \cup \\ & \{2A(n_1, p) + 2A(n_2, p - 1) + \cdots + 2A(n_{p-3}, 4) + A(n_{p-2}, 3) \mid \\ & \qquad n_1 + \cdots + n_{p-2} = n, n_1 \geq n_2 \geq \cdots\} \cup \cdots \cup \\ & \{2A(n_1, p) + A(n_2, p - 1) \mid n_1 + n_2 = n, n_1 \geq n_2\}, \end{aligned}$$

where $n_i \in \mathbb{N}$.

3 Proof of $A' = F' = S$

We first state the following two lemmas on the function $A'(n, p)$. Even if the first one is more or less obvious (from the fact that $A'(n, p) \geq M(n, p) \geq 2n - 1$) and the second deals basically only with the trivial cases, we include its formal proof, the reason being that several “obvious facts” related to the Tower of Hanoi problem turned out to be false.

Lemma 3.1 For any $n \geq 1, p \geq 3$ we have $A'(n, p) \geq 2n - 1$.

Proof. For $n < p$ lemma holds by the definition of $A'(n, p)$. For $p = 3$ and $n \geq 3$ we have $A'(n, 3) = 2^n - 1 \geq 2n - 1$. Finally, for $p \geq 4$ and $n \geq p$ we

have

$$A'(n, p) = 2A'(n_1, p) + 2A'(n_2, p-1) + \cdots + 2A'(n_{p-i}, i+1) + A'(n_{p-i+1}, i),$$

where $n_1 + n_2 + \cdots + n_{p-i+1} = n$ and $2 \leq i \leq p-2$. By induction we have

$$\begin{aligned} A'(n, p) &\geq 2[(2n_1 - 1) + \cdots + (2n_{p-i} - 1)] + 2n_{p-i+1} - 1 \\ &= 2(n_1 + \cdots + n_{p-i} + n_{p-i+1}) + 2(n_1 + \cdots + n_{p-i}) - 2(p-i) - 1 \\ &= 2n + 2(n_1 + \cdots + n_{p-i}) - 2(p-i) - 1 \geq 2n - 1, \end{aligned}$$

and we are done. \square

Lemma 3.2 *For any $n \geq 1$, $p \geq 3$ and i, n_1, \dots, n_{p-i+1} with $2 \leq i \leq p-1$ and $n = n_1 + n_2 + \cdots + n_{p-i+1}$ we have*

$$A'(n, p) \leq 2A'(n_1, p) + 2A'(n_2, p-1) + \cdots + 2A'(n_{p-i}, i+1) + A'(n_{p-i+1}, i).$$

Proof.

If $p \geq 4$ and $n \geq p$ the statement follows directly from the definition.

For $p = 3$ it follows $i = 2$ and we have to prove only $A'(n, 3) \leq 2A'(n_1, 3) + A'(n_2, 2)$, which is nondegenerate only for $n_2 = 1$. But then we have to prove $A'(n, 3) \leq 2A'(n-1, 3) + A'(1, 2)$, which is trivial (since it says $2^n - 1 \leq 2(2^{n-1} - 1) + 1$).

The final case is when $n < p$, $p \geq 4$, is done by induction on n .

If $n = 2$ then $2 = n = n_1 + n_2 + \cdots + n_{p-i+1} \geq p-i+1$. Thus $p-i \leq 1$ and as on the other hand we have $p-i \geq 1$ it follows that $p-i = 1$. Then $n_1 = n_2 = 1$. Therefore one only has to show that $A'(2, p) \leq 2A'(1, p) + A'(1, p-1)$, which is indeed the case ($3 \leq 2 \cdot 1 + 1$).

The inductive step is as follows:

$$\begin{aligned} &2A'(n_1, p) + \left(2A'(n_2, p-1) + \cdots + 2A'(n_{p-i}, i+1) + A'(n_{p-i+1}, i)\right) \\ &\geq 2A'(n_1, p) + A'(n - n_1, p-1). \end{aligned}$$

Therefore, our claim will be proved by showing that $2A'(n_1, p) + A'(n - n_1, p-1) \geq A'(n, p)$. By Lemma 3.1 we have

$$\begin{aligned} 2A'(n_1, p) + A'(n - n_1, p-1) &\geq 2(2n_1 - 1) + 2(n - n_1) - 1 \\ &= 2(n_1 + n - n_1) + (2n_1 - 3) \geq 2n - 1 = A'(n, p), \end{aligned}$$

which completes the argument. \square

Proposition 3.3 *For any $n \geq 1$, $p \geq 3$ we have $A'(n, p) = S(n, p)$.*

Proof. The proof is by induction on n . More precisely, we are going to prove that the claim holds for all $n \geq 4$, and, for a fixed n , that it is true for all p with $4 \leq p \leq n$.

For $n = 4$ we only need to consider $p = 4$. A direct simple computation shows that in this case we have

$$A'(4, 4) = S(4, 4) = 9.$$

Assume now that the induction hypothesis is true for all m with $4 \leq m < n$. Let p be an arbitrary integer satisfying $4 \leq p \leq n$.

The induction hypothesis implies that for all $n_1, n_2 \in \mathbb{N}$ for which $n_1 + n_2 = n$ we have

$$2S(n_1, p) + S(n_2, p - 1) = 2A'(n_1, p) + A'(n_2, p - 1).$$

Thus, the set by which we define $S(n, p)$ (as its minimum) is a subset of the set by which we define $A'(n, p)$ (as its minimum). It follows that $A'(n, p) \leq S(n, p)$.

It remains to prove that $A'(n, p) \geq S(n, p)$. If for some n_1 and n_2 with $n_1 + n_2 = n$ we have $A'(n, p) = 2A'(n_1, p) + A'(n_2, p - 1)$, then we are done, since in this case $A'(n, p)$ is in the set of which $S(n, p)$ is the minimum. Suppose therefore that

$$A'(n, p) = 2A'(n_1, p) + 2A'(n_2, p - 1) + \cdots + 2A'(n_{p-i}, i + 1) + A'(n_{p-i+1}, i),$$

where $n_1 + n_2 + \cdots + n_{p-i+1} = n$ and $2 \leq i \leq p - 2$. (Note that in the particular case of $i = 2$ we have $n_{p-i+1} = n_{p-1} = 1$ and $A'(n_{p-1}, 2) = 1$.) Then $n_2 + n_3 + \cdots + n_{p-i+1} = n - n_1$. By Lemma 3.2 we have

$$A'(n - n_1, p - 1) \leq 2A'(n_2, p - 1) + \cdots + 2A'(n_{p-i}, i + 1) + A'(n_{p-i+1}, i),$$

and therefore, using the induction hypothesis, we conclude

$$A'(n, p) \geq 2A'(n_1, p) + A'(n - n_1, p - 1) = 2S(n_1, p) + S(n - n_1, p - 1) \geq S(n, p).$$

\square

Proposition 3.4 *For any $n \geq 1$ and any $p \geq 3$ we have $A'(n, p) = F'(n, p)$.*

Proof. As in the proof of Proposition 3.3 we proceed by induction on n . For $n = 4$ we have:

$$\begin{aligned}
F'(4, 4) &= \min\{2F'(n_1, 4) + 2F'(n_2, 3) + 1 \mid n_1 + n_2 + 1 = 4\} \\
&= \min\{2A'(n_1, 4) + 2A'(n_2, 3) + 1 \mid n_1 + n_2 = 3\} \\
&= \min\{2A'(1, 4) + 2A'(2, 3) + 1, 2A'(2, 4) + 2A'(1, 3) + 1\} \\
&= \min\{2(1 + 3) + 1, 2(3 + 1) + 1\} = 9 = A'(4, 4).
\end{aligned}$$

Assume now that $F'(m, p) = A'(m, p)$ for all m with $4 \leq m < n$. The set by which we define $F'(n, p)$ (as its minimum) is a subset of the set by which we define $A'(n, p)$ (as its minimum), because for all n_1, \dots, n_{p-2} for which $n_1 + \dots + n_{p-2} + 1 = n$ we have

$$2F'(n_1, p) + \dots + 2F'(n_{p-2}, 3) + 1 = 2A'(n_1, p) + \dots + 2A'(n_{p-2}, 3) + 1.$$

Therefore, $F'(n, p) \geq A'(n, p)$.

To prove the other inequality, we use the following dynamic programming argument. By Proposition 3.3 we can write

$$A'(n, p) = 2A'(n_1, p) + A'(n_2^0, p - 1),$$

where $n_1 + n_2^0 = n$. Applying this argument again we have $A'(n_2^0, p - 1) = 2A'(n_2, p - 1) + A'(n_3^0, p - 2)$, where $n_2 + n_3^0 = n_2^0$ and so

$$A'(n, p) = 2A'(n_1, p) + 2A'(n_2, p - 1) + A'(n_3^0, p - 2).$$

Now, how far can this procedure be executed? We distinguish two cases.

Case 1. $A'(n, p) = 2A'(n_1, p) + \dots + 2A'(n_{p-i}, i + 1) + A'(n_{p-i+1}^0, i)$, with $i > 2$ and $n_{p-i+1}^0 = 1$.

Set $r = \max\{k \mid n_k > 1\}$ and note that $r \leq p - i$. Then we have:

$$\begin{aligned}
A(n, p) &= 2A(n_1, p) + \dots + 2A(n_r, p - r + 1) + 2A(1, p - r) + \\
&\quad \dots + 2A(1, i + 1) + 2A(1, i) \\
&= 2A(n_1, p) + \dots + 2\left(2A(n'_r, p - r - 1) + A(n'_{r+1}, p - r)\right) + \\
&\quad 2A(1, p - r - 1) + \dots + 2A(1, i) + A(1, i - 1) \\
&> 2A(n_1, p) + \dots + 2A(n'_r, p - r + 1) + 2A(n'_{r+1}, p - r) + \\
&\quad 2A(1, p - r - 1) + \dots + 2A(1, i) + A(1, i - 1) \geq A(n, p).
\end{aligned}$$

Note that in the above computation the change of variables in the second equality is possible since all the values of $A'(1, p - r), \dots, A'(1, i)$ as well of $A'(1, p - r - 1), \dots, A'(1, i - 1)$ are equal to 1 and $i - 1 \geq 2$. Thus we have a contradiction which shows that this case is not possible.

Case 2. $A'(n, p) = 2A'(n_1, p) + \cdots + 2A'(n_{p-i}, i+1) + A'(n_{p-i+1}^0, i)$, with $i = 2$ and $n_{p-i+1}^0 = 1$.

Now we have

$$\begin{aligned} A'(n, p) &= 2A'(n_1, p) + \cdots + 2A'(n_{p-2}, 3) + 1 \\ &= 2F'(n_1, p) + \cdots + 2F'(n_{p-2}, 3) + 1 \\ &\geq F'(n, p) \end{aligned}$$

and we are done. \square

Combining Propositions 3.3 and 3.4 we can state:

Corollary 3.5 *For any $n \geq 1$ and any $p \geq 3$ we have*

$$A'(n, p) = F'(n, p) = S(n, p) \leq M(n, p).$$

4 Explicit formulas

Explicit formulas we are going to discuss in this section had appeared already in Frame [6], but have been treated rather heuristically, and — which seems to be the major deficiency of Frame's approach — they appeared as statements about $M(n, p)$. Therefore, we believe it is a necessity to give an independent treatment of them, and using only their properties as well as properties of A', F', S , to prove they really represent these functions (that shall be done in the following section — and it will turn out to be rather nontrivial).

Definition 4.1 *Let $n \geq 2$ and $p \geq 3$. Then $X(n, p)$ is defined as*

$$X(n, p) = \sum_{t=0}^{s-1} 2^t \frac{(p-3+t)!}{(p-3)!t!} + 2^s \left[n - \frac{(p-3+s)!}{(p-2)!(s-1)!} \right],$$

where s is the largest integer for which the last term on the right side is positive, i. e.

$$s = \max \left\{ k \in \mathbb{Z} \mid n - \frac{(p-3+k)!}{(p-2)!(k-1)!} > 0 \right\}.$$

$X(1, p)$ is defined as

$$X(1, p) = 1.$$

Using binomial coefficients notation, this definition (in case $n \geq 2$) can be reformulated in the following way:

$$X(n, p) = \sum_{t=0}^{s-1} 2^t \binom{p-3+t}{p-3} + 2^s \left[n - \binom{p-3+s}{p-2} \right],$$

where

$$s = \max \left\{ k \in \mathbb{Z} \mid n - \binom{p-3+k}{p-2} > 0 \right\}.$$

Already this is a sufficient motivation for the introduction of functions $h_p(x)$, $p \in \mathbb{N}$:

Definition 4.2 For $p \geq 3$, let

$$h_p(x) = \binom{p-3+x}{p-2}, \quad x \in \mathbb{R}, \quad x \geq 0.$$

From their direct presentation in the form

$$h_p(x) = \frac{x(x+1) \cdots (x+p-4)(x+p-3)}{(p-2)!}$$

it is obvious that (for $p \geq 3$) these functions are strictly increasing polynomials (on nonnegative reals).

Since they are strictly increasing, they have inverses

$$g_p = h_p^{-1}$$

which are strictly increasing on nonnegative reals as well.

Definition 4.3 Define (for $p \geq 3$)

$$f_p(x) = \lceil g_p(x) \rceil,$$

where $g_p = h_p^{-1}$.

Lemma 4.4 The number s from Definition 4.1 (for $n \geq 2$) can be written as

$$s = f_p(n) - 1.$$

Proof. Obviously

$$s = \max \{ k \in \mathbb{Z} \mid n - h_p(k) > 0 \}.$$

Since h_p is strictly increasing, this means that s is determined as the unique integer satisfying

$$h_p(s) < n, \quad h_p(s+1) \geq n$$

or equivalently

$$s < g_p(n), \quad s+1 \geq g_p(n),$$

i.e.

$$g_p(n) \leq s+1 < g_p(n)+1,$$

and that proves the lemma. \square

By Lemma 4.4 $X(n, p)$ (for $n \geq 2$) may be described by a single formula as follows:

$$X(n, p) = \sum_{t=0}^{f_p(n)-2} 2^t \binom{p-3+t}{p-3} + 2^{f_p(n)-1} \left[n - \binom{p+f_p(n)-4}{p-2} \right].$$

Lemma 4.5 $X(n, p) - X(n-1, p) = 2^{f_p(n)-1}$, for any $n \geq 2$ and $p \geq 3$.

Proof. $X(2, p) - X(1, p) = 2^{f_p(2)-1}$ is true, because $f_p(2) = 2$. If $n \geq 3$, two cases are distinguished.

Case 1. $f_p(n-1) = f_p(n)$.

In this case, $X(n, p) - X(n-1, p)$ equals to

$$2^{f_p(n)-1} \left[n - \binom{p+f_p(n)-4}{p-2} \right] - 2^{f_p(n)-1} \left[n-1 - \binom{p+f_p(n)-4}{p-2} \right],$$

i.e. to $2^{f_p(n)-1}(n - (n-1)) = 2^{f_p(n)-1}$.

Case 2. $f_p(n-1) \neq f_p(n)$.

If we denote $f_p(n) - 1$ by s , then $s = g_p(n-1)$. Suppose not. Then it would be $g_p(n-1) < s$, hence $n-1 < h_p(s)$. From $f_p(n) = s+1$ it follows $g_p(n) \geq s+1$ and hence $n \geq h_p(s+1)$. So, it would follow that the integer $h_p(s+1) - h_p(s)$ is strictly between 0 and 1 — and it's a contradiction.

Obviously, it follows also that $f_p(n-1) = s$.

Therefore, in this case $X(n, p) - X(n-1, p)$ can be written as

$$2^{f_p(n)-2} \binom{p+f_p(n)-5}{p-3} + 2^{f_p(n)-1} \left[n - \binom{p+f_p(n)-4}{p-2} \right] - 2^{f_p(n-1)-1} \left[n-1 - \binom{p+f_p(n-1)-4}{p-2} \right].$$

This simplifies to

$$\begin{aligned}
& 2^{s-1} \binom{p+s-4}{p-3} + 2^s \left[n - \binom{p+s-3}{p-2} \right] - 2^{s-1} \left[n-1 - \binom{p+s-4}{p-2} \right] = \\
& = 2^{s-1} \left[\binom{p+s-4}{p-3} - 2 \binom{p+s-3}{p-2} + \binom{p+s-4}{p-2} \right] + 2^{s-1} [2n - n + 1] = \\
& = 2^{s-1} \left[n + 1 - \binom{p+s-3}{p-2} \right] = 2^s,
\end{aligned}$$

since

$$\binom{p+s-3}{p-2} = n-1.$$

□

Summing up these equalities, together with $X(1, p) = 1 = 2^{f_p(1)-1}$, we obtain

Theorem 4.6 *For any $n \geq 1$ and $p \geq 3$*

$$X(n, p) = \sum_{k=1}^n 2^{f_p(k)-1}.$$

5 Proof of $X = S$

In this section we prove the following theorem:

Theorem 5.1 *For all $p \geq 3$, $n \geq 1$, it holds true that*

$$X(n, p) = S(n, p).$$

Proof.

We shall prove the claim inductively, together with the following statement:

$$S(h_p(k), p) = 2S(h_p(k-1), p) + S(h_p(k) - h_p(k-1), p-1),$$

for $p \geq 4$, and $k \geq 2$.

This additional claim forms an important ingredient of our inductive proof of the original statement, but also, together with presentations of $S(n, p)$ in the form $2S(n_1, p) + S(n_2, p-1)$ given in the proof, it can be used for computational purposes.

Note that $h_p(k) - h_p(k-1) = h_{p-1}(k)$ — what we have here are binomial coefficients in disguise!

The trivial range is settled easily.

$$S(n, 3) = 2^n - 1 = 1 + 2 + \dots + 2^{n-1} = X(n, 3), \text{ since } f_3(k) = k.$$

$$\text{Also, for } p \geq 4 \text{ and } n < p, \text{ we have } X(n, p) = 1 + (n-1)2 = 2n - 1 = S(n, p).$$

$$\text{In addition to that, we see that } 2S(h_p(1), p) + S(h_{p-1}(2), p-1) = 2 \cdot 1 + 2(p-2) - 1 = 2p - 3 = 2(p-1) - 1 = S(h_p(2), p).$$

Now, let n satisfy the inequality

$$h_p(k) + 1 \leq n \leq h_p(k+1),$$

for some $k \geq 2$. Our inductive assumption is that $S(m, p) = X(m, p)$ for all m , $1 \leq m < n$, and that $S(h_p(k), p) = 2S(h_p(k-1), p) + S(h_{p-1}(k), p-1)$.

Then $f_p(n) = k+1$, $f_p(h_p(k)) = k$, and $f_p(\ell) \geq k+2$, for all $\ell > h_p(k+1)$.

If $n_1 > h_p(k)$, then $f_p(n_1) > k$, and therefore

$$\begin{aligned} 2S(n_1, p) + S(n - n_1, p - 1) &= 2X(n_1, p) + X(n - n_1, p - 1) = \\ &= 2\left(X(n_1 - 1, p) + 2^{f_p(n_1)-1}\right) + X(n - n_1, p - 1) > \\ &= 2S(n_1 - 1, p) + S(n - n_1, p - 1) + 2^k = S(n - 1, p) + 2^k = \\ &= S(n - 1, p) + 2^{f_p(n)-1} = X(n, p). \end{aligned}$$

If $n_1 \leq h_p(k-1)$, then $n - n_1 \geq n - h_p(k-1) > h_p(k) - h_p(k-1) = h_{p-1}(k)$, and therefore $f_{p-1}(n - n_1) \geq k+1$. It follows

$$\begin{aligned} 2S(n_1, p) + S(n - n_1, p - 1) &= 2X(n_1, p) + X(n - n_1, p - 1) \\ &= X(n_1, p) + 2X(n - n_1 - 1, p - 1) + 2^{f_{p-1}(n-n_1)-1} \geq \\ &= S(n - 1, p) + 2^k = X(n, p). \end{aligned}$$

In the remaining case $h_p(k-1) < n \leq h_p(k)$, since $f_p(n_1) = k$, we have

$$\begin{aligned} 2S(n_1, p) + S(n - n_1, p - 1) &= 2X(n_1, p) + X(n - n_1, p - 1) = \\ &= 2\left(X(n_1 - 1, p) + 2^{f_p(n_1)-1}\right) + X(n - n_1, p - 1) = \\ &= 2S(n_1 - 1, p) + S(n - n_1, p - 1) + 2^k \geq S(n - 1, p) + 2^k = X(n, p). \end{aligned}$$

Thus, since these three cases cover all the possibilities, it follows that $S(n, p) \geq X(n, p)$. Also, if some n_1 satisfies $2S(n_1, p) + S(n - n_1, p - 1) = X(n, p)$, it means that $S(n, p) = X(n, p) = 2S(n_1, p) + S(n - n_1, p - 1)$.

Suppose $S(n-1, p) = X(n-1, p) = 2S(m_1, p) + S(n-1-m_1, p-1)$, where $h_p(k-1) \leq m_1$, $m_1+1 \leq h_p(k)$. In that case $f_p(m_1+1) = k$ and

$$\begin{aligned} 2S(m_1+1, p) + S(n-(m_1+1), p-1) &= \\ 2\left(S(m_1, p) + 2^{f_p(m_1+1)-1}\right) + S(n-(m_1+1), p-1) &= \\ 2S(m_1, p) + S(n-1-m_1, p-1) + 2^k &= S(n-1, p) + 2^k = \\ X(n-1, p) + 2^{f_p(n)-1} &= X(n, p) \end{aligned}$$

and it follows that $S(n, p) = X(n, p) = 2S(m_1+1, p) + S(n-1-m_1, p-1)$.

Using this result, starting from inductive assumption

$$S(h_p(k), p) = X(h_p(k), p) = 2S(h_p(k-1), p) + S(h_p(k) - h_p(k-1), p-1)$$

we get³

$$\begin{aligned} S(h_p(k)+1, p) &= X(h_p(k)+1, p) = \\ &\quad 2S(h_p(k-1)+1, p) + S(h_p(k) - h_p(k-1), p-1), \\ S(h_p(k)+2, p) &= X(h_p(k)+2, p) = \\ &\quad 2S(h_p(k-1)+2, p) + S(h_p(k) - h_p(k-1), p-1), \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ S(m, p) &= X(m, p) = \\ &\quad 2S(h_p(k), p) + S(h_p(k) - h_p(k-1), p-1), \end{aligned}$$

where $m = h_p(k) + (h_p(k) - h_p(k-1))$.

It is obvious that $m < h_p(k+1)$ (since it is equivalent to $h_p(k) - h_p(k-1) < h_p(k+1) - h_p(k)$, i.e. to $h_{p-1}(k) < h_{p-1}(k+1)$).

Hence, it remains to fulfill the inductive step in the remaining case when n satisfies $m < n \leq h_p(k+1)$. “Inductive proof” now means that inductively we prove $S(t, p) = X(t, p) = 2S(h_p(k), p) + S(t - h_p(k))$ for all t satisfying $m \leq t \leq h_p(k+1)$. Note that that is true for $t = m$, by the choice of m .

Let n satisfies $m < n \leq h_p(k+1)$ and let the assumption holds true for $n-1$.

In that case $h_p(k) - h_p(k-1) = m - h_p(k) < n - h_p(k) \leq h_p(k+1) - h_p(k)$. It means $h_{p-1}(k) < n - h_p(k) \leq h_{p-1}(k+1)$, and therefore it follows that $f_{p-1}(n - h_p(k)) = k+1$.

Now we have

³ In [7] Hinz uses a similar approach for the proof of $X(n, 4) = S(n, 4)$.

$$\begin{aligned}
& 2S(h_p(k), p) + S(n - h_p(k), p - 1) = \\
& 2X(h_p(k), p) + X(n - h_p(k), p - 1) = \\
& 2X(h_p(k), p) + X(n - 1 - h_p(k), p - 1) + 2^{f_{p-1}(n-h_p(k))-1} = \\
& X(n - 1, p) + 2^k = X(n, p).
\end{aligned}$$

Note that as a special case (for $n = h_p(k + 1)$) we get

$$2S(h_p(k), p) + S(h_p(k + 1) - h_p(k), p - 1) = S(h_p(k + 1), p),$$

thus finishing also the inductive proof of the additional claim, introduced at the beginning of the proof. \square

6 And finally: All together now

Finally, we are able to show that the original Frame's function F , as well as our function A , both coincide with all the rest.

Lemma 6.1 $X(n, p) - X(n - 1, p) \leq X(n, p - 1) - X(n - 1, p - 1)$.

Proof. It means $2^{f_p(n)-1} \leq 2^{f_{p-1}(n)-1}$, hence is equivalent to $f_p(n) \leq f_{p-1}(n)$, which is easily seen to be true. \square

Corollary 6.2 $X(m, p) + X(n, p - 1) \geq X(n, p) + X(m, p - 1)$, if $m < n$.

Proof. The inequality is equivalent to

$$X(n, p) - X(m, p) \leq X(n, p - 1) - X(m, p - 1)$$

and this follows by summing up inequalities

$$\begin{aligned}
& X(n, p) - X(n - 1, p) \leq X(n, p - 1) - X(n - 1, p - 1) \\
& X(n - 1, p) - X(n - 2, p) \leq X(n - 1, p - 1) - X(n - 2, p - 1) \\
& \quad \cdot \\
& \quad \cdot \\
& \quad \cdot \\
& X(m + 2, p) - X(m + 1, p) \leq X(m + 2, p - 1) - X(m + 1, p - 1) \\
& X(m + 1, p) - X(m, p) \leq X(m + 1, p - 1) - X(m, p - 1)
\end{aligned}$$

which hold by Lemma 6.1. \square

Proposition 6.3 For any $n \geq 1$ and any $p \geq 3$ we have $F'(n, p) = F(n, p)$.

Proof. For the trivial range it is so by definition. Therefore we shall prove the equality by induction on n for $p \geq 4$ and $n \geq p$.

Obviously, $F'(n, p) \leq F(n, p)$, since more partitions are taken into account on the left hand side.

Let $F'(n, p) = 2F'(n_1, p) + \cdots + 2F'(n_{p-2}, 3) + 1$, where $n_1 + \cdots + n_{p-2} + 1 = n$ and $n_i < n_{i+1}$, for some i .

Then, using Corollary 6.2 and inductive assumptions, we get

$$\begin{aligned}
F'(n, p) &= \\
&2\left(F(n_1, p) + \cdots + F(n_{i-1}, p - i + 2) + \right. \\
&\quad \left. F(n_i, p - i + 1) + F(n_{i+1}, p - i) + \cdots\right) + 1 = \\
&2\left(X(n_1, p) + \cdots + X(n_{i-1}, p - i + 2) + \right. \\
&\quad \left. X(n_i, p - i + 1) + X(n_{i+1}, p - i) + \cdots\right) + 1 \geq \\
&2\left(X(n_1, p) + \cdots + X(n_{i-1}, p - i + 2) + \right. \\
&\quad \left. X(n_{i+1}, p - i) + X(n_i, p - i + 1) + \cdots\right) + 1 \geq \\
&2\left(F(n_1, p) + \cdots + F(n_{i-1}, p - i + 2) + \right. \\
&\quad \left. F(n_{i+1}, p - i) + F(n_i, p - i + 1) + \cdots\right) + 1.
\end{aligned}$$

After finitely many repetitions of this procedure, we get that $F'(n, p)$ is at least as large as one of the defining sums for $F(n, p)$, hence $F'(n, p) \geq F(n, p)$. \square

Proposition 6.4 *For any $n \geq 1$ and any $p \geq 3$ we have $A'(n, p) = A(n, p)$.*

Proof. The trivial range is once again trivial. In the nontrivial range we have $A'(n, p) \leq A(n, p)$, since the subset defining the right hand side is a subset of the set defining the left hand side. Then, we have $A(n, p) \leq F(n, p)$, for the same reason, and we already know that $A'(n, p) = F(n, p)$. \square

Hence, we have completed the proof of

Theorem 6.5 *For any $n \geq 1$ and any $p \geq 3$ we have*

$$A(n, p) = A'(n, p) = F(n, p) = F'(n, p) = S(n, p) = X(n, p) \geq M(n, p).$$

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